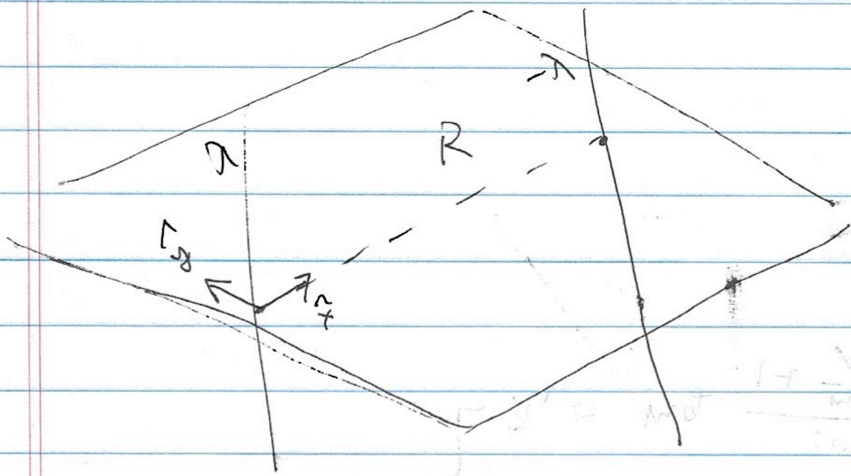
Jackson 2.8 (a)



The potential distance r away from line charge is

$$\Phi = -\frac{\lambda}{2\pi\varepsilon_0}\ln r.$$

$$\Rightarrow \quad \Phi(x,y) = -\frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{x^2+y^2} + \frac{\lambda}{2\pi\varepsilon_0}\ln\sqrt{(R-x)^2+y^2} = V$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\ln\sqrt{(R-x)^2+y^2} - \ln\sqrt{x^2+y^2}\right]$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\ln\left[\frac{\sqrt{(R-x)^2+y^2}}{\sqrt{x^2+y^2}}\right] = V.$$

$$\ln\frac{\sqrt{(R-x)^2+y^2}}{\sqrt{x^2+y^2}} = \frac{2\pi\varepsilon_0 V}{\lambda}.$$

$$\frac{1}{2}\ln\frac{(R-x)^2+y^2}{x^2+y^2} = \frac{2\pi\varepsilon_0 V}{\lambda}$$

$$\frac{(R-x)^2+y^2}{x^2+y^2} = \exp\left[\frac{4\pi\varepsilon_0 V}{\lambda}\right]$$

Let $\exp\left[\frac{4\pi\varepsilon_0 V}{\lambda}\right] = C$, we have equation of motion

$$\frac{(R-x)^2+y^2}{x^2+y^2} = C$$

$$R^2+x^2-2Rx+y^2-Cx^2-Cy^2=0$$

$$(1-C)x^2-2Rx+R^2+(1-C)y^2=0.$$

$$x^2-\frac{2R}{1-C}x+\frac{R^2}{1-C}+y^2=0$$

$$y^2+\left(x-\frac{R}{1-C}\right)^2-\frac{R^2}{(1-C)^2}+\frac{R^2}{1-C}=0$$

This is equation of a circle:

$$y^2+\left(x-\frac{R}{1-C}\right)^2=\left[\frac{R^2}{(1-C)^2}-\frac{R^2}{1-C}\right]$$

The radius is $\left[\frac{R^2}{(1-C)^2}-\frac{R^2}{1-C}\right]^{1/2}$

The coordinate is $x=\frac{R}{1-C}$, $y=0$.

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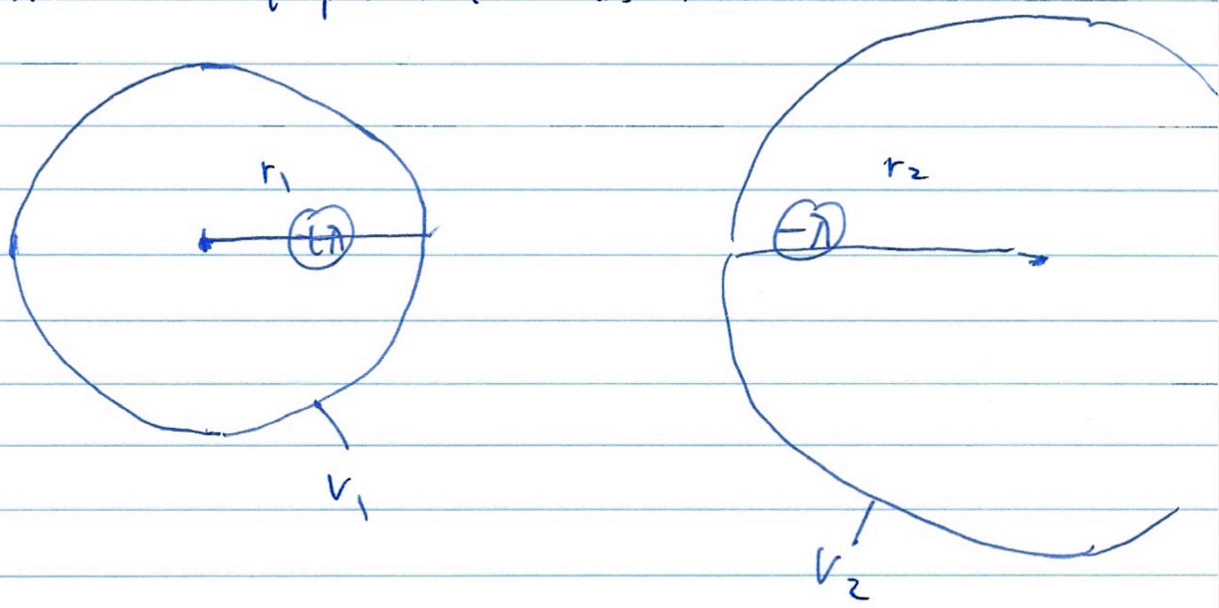
Jackson 2.8 (b)

The coordinate and radius for the equipotential circle is

$$x = \frac{R}{1-c}, \quad y = 0, \qquad r = R\left[\frac{1}{(1-c)^2} - \frac{1}{1-c}\right]^{1/2}$$

where $$c = \exp\left[\frac{4\pi\varepsilon_0 V}{\lambda}\right]$$

Consider 2 equipotential circles illustrated as below



$$r_1 = R\left[\frac{1}{(1-c_1)^2} - \frac{1}{1-c_1}\right]^{1/2} = a$$

$$r_2 = R\left[\frac{1}{(1-c_2)^2} - \frac{1}{1-c_2}\right]^{1/2} = b$$

$$x_2 - x_1 = \frac{R}{1-c_2} - \frac{R}{1-c_1} = d$$

$$a^2 = R^2\left[\frac{1}{(1-c_1)^2} - \frac{1}{1-c_1}\right], \quad b^2 = R^2\left[\frac{1}{(1-c_2)^2} - \frac{1}{1-c_2}\right]$$

$$d^2 = R^2\left[\frac{1}{(1-c_1)^2} + \frac{1}{(1-c_2)^2} - \frac{2}{(1-c_1)(1-c_2)}\right]$$

$$a^2+b^2 = R^2\left[\frac{1}{(1-c_1)^2} + \frac{1}{(1-c_2)^2} - \frac{1}{1-c_1} - \frac{1}{1-c_2}\right]$$

$$= R^2\left[\frac{1}{(1-c_1)^2} + \frac{1}{(1-c_2)^2} - \frac{2-c_2-c_1}{(1-c_1)(1-c_2)}\right]$$

$$\Rightarrow d^2-(a^2+b^2) = R^2\left[\frac{c_1+c_2}{(1-c_1)(1-c_2)}\right]$$

R^2 can be eliminated via ab :

$$ab = R^2\sqrt{\left(\frac{1}{(1-c_1)^2} - \frac{1}{1-c_1}\right)\left(\frac{1}{(1-c_2)^2} - \frac{1}{1-c_2}\right)}$$

$$= R^2\left[\frac{1-(1-c_2)-(1-c_1)+(1-c_1)(1-c_2)}{(1-c_1)^2(1-c_2)^2}\right]^{1/2}$$

$$= R^2\frac{\sqrt{c_1c_2}}{(1-c_1)(1-c_2)}$$

It's then tempting to write

$$\frac{d^2-(a^2+b^2)}{ab} = \cancel{R^2}\left[\frac{c_1+c_2}{\cancel{(1-c_1)(1-c_2)}}\right]\frac{\cancel{(1-c_1)(1-c_2)}}{\cancel{R^2}\sqrt{c_1c_2}}$$

$$= \frac{c_1+c_2}{\sqrt{c_1c_2}}$$

Recall $c_1 = \exp\left[\frac{4\pi\varepsilon_0 V_1}{\lambda}\right]$, plug this in:

$$\frac{d^2-(a^2+b^2)}{ab} = \frac{\exp\left[\frac{4\pi\varepsilon_0}{\lambda}V_1\right]+\exp\left[\frac{4\pi\varepsilon_0}{\lambda}V_2\right]}{\exp\left[\frac{2\pi\varepsilon_0}{\lambda}(V_1+V_2)\right]}$$

$$= \exp\left[\frac{2\pi\varepsilon_0}{\lambda}(V_1-V_2)\right]+\exp\left[\frac{2\pi\varepsilon_0}{\lambda}(V_2-V_1)\right]$$

Then it's evident that

$$\cosh^{-1}\left[\frac{d^2-(a^2+b^2)}{2ab}\right] = \frac{2\pi\varepsilon_0}{\lambda}(V_1-V_2)$$

$$= \frac{2\pi\varepsilon_0}{\lambda}\left[V_{diff}\right]$$

The capacitance formula $C = \frac{Q}{V}$ gives the answer.

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Jackson 2.8 (c)

The appropriate limit is $d \gg a, b$

then $\dfrac{d^2 - a^2 - b^2}{2ab} \approx \dfrac{d^2}{2ab}$

For large θ, $\cosh^{-1}\theta \approx \ln 2\theta$

$$\Rightarrow \cosh^{-1}\left[\frac{d^2 - a^2 - b^2}{2ab}\right] \approx \ln\left[\frac{d^2}{ab}\right]$$

$$= 2\ln\left[\frac{d}{\sqrt{ab}}\right]$$

Then $C = \dfrac{2\pi\varepsilon_0}{\cosh^{-1}[\ldots]}$ reduces to

$$C \approx \frac{2\pi\varepsilon_0}{2\ln\left[\frac{d}{\sqrt{ab}}\right]}$$

$$= \pi\varepsilon_0\left[\ln\frac{d}{\sqrt{ab}}\right]^{-1}$$

which is the formula from exercise 1.7.

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